The Fundamental Theorem of Calculus:

Two major components of Calculus are

① Differentiations – study of tangents

② Integrals – study of area under curves.

Q:- Is there any way to relate these two components?

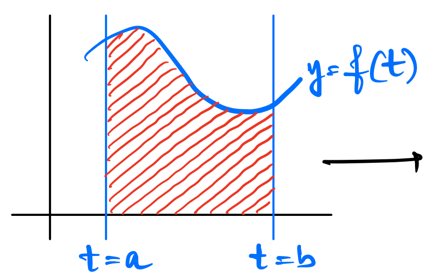
Ans:- YES

Let's understand how!!

We start with

$$\int_a^b f(t)\,dt =$$

a continuous function

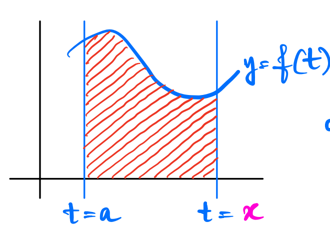


→ gives a number

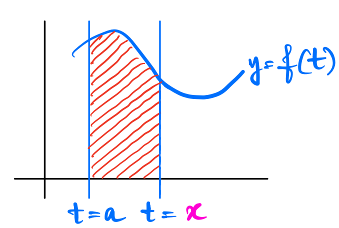
where a, b are fixed.

Now we keep a fixed & choose b to be 'x', a variable.
Then it's like end t = a is fixed & t = b is a slider.

$$\int_a^x f(t)\,dt =$$



or



in this case, it's not just a number, it's an area function, depending on x.

So, let

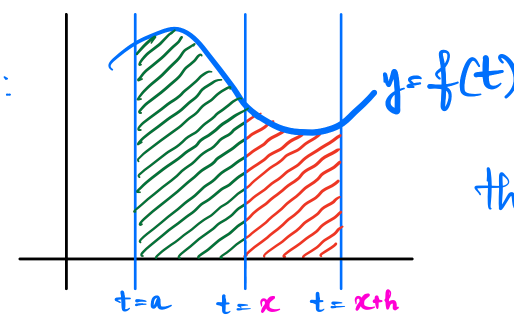
$$F(x) = \int_a^x f(t)\,dt \rightsquigarrow \text{Accumulation function.}$$

Now what happens if we try computing $F'(x)$?

By definition, $F'(x) = \lim_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$

Here, $F(x+h) = \int_a^{x+h} f(t)\,dt$

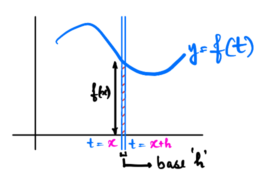
Now, $F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$

geometrically: 

the area in red.

$$= \int_x^{x+h} f(t)\,dt.$$

So, $\dfrac{F(x+h) - F(x)}{h} = \dfrac{\int_x^{x+h} f(t)\,dt}{h}$

Now, when $h \to 0$ we get 

$$\therefore \underbrace{\lim_{h\to 0} \frac{F(x+h) - F(x)}{h}}_{= F'(x)} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0} \frac{\text{Area of a rectangle of base } h \text{ \& height } f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot h}{h} = \lim_{h\to 0} f(x) = f(x)$$

Formal Statement :- Suppose f is a continuous function on $[a,b]$.

Let $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$

or

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

In other words: Every continuous function f possesses an anti-derivative, namely

$$F(x) = \int_a^x f(t)\,dt$$

Another form of the Fundamental Theorem of Calculus (FTC):

$$\int_a^b f(t)\,dt = F(b) - F(a),$$

where, $F(x) = \int_a^x f(t)\,dt$ is known.

Example :- ① Find the derivative of $\int_0^x \sqrt{t^2+1}\,dt$

$$\frac{d}{dx}\int_0^x \sqrt{t^2+1}\,dt = \sqrt{x^2+1}$$

② For $f(x) = x^2$, $F(x) = \frac{x^3}{3}$ is known.

$$\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \underbrace{\left(\frac{2^3}{3}\right)}_{F(a)} - \underbrace{\left(\frac{0^3}{3}\right)}_{F(b)}$$

$$= \frac{8}{3} - 0 = \frac{8}{3}$$